NUMERICAL SOLUTION OF RETROSPECTIVE HEAT-CONDUCTION PROBLEMS

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The article presents a method of solving heat-conduction problems by determining the initial temperature field or the instant when some physical event occurred.

The inverse heat-conduction problems (IHCP), which are called retrospective or reversed [1-3], include problems connected with the determination of the initial temperature field or the instant when, e.g., an event began or ended, which may have been a phase transformation, a given temperature that was attained at some point of a body, etc.

In formulating retrospective IHCP some additional information ensuring an unambiguous solution of the IHCP is specified instead of unknown initial conditions that are indispensable for solving direct heat-conduction problems; such information is usually either a temperature field some time after the initial instant, or a change of temperature or of heat flux at one or several points of a body in the course of a certain time interval.

We will examine the following one-dimensional heat-conduction problem:

$$c\rho \frac{\partial t}{\partial \tau} = \frac{1}{r^{\Gamma}}\left[\frac{\partial}{\partial r}\left(r^{\Gamma}\lambda \frac{\partial t}{\partial r}\right)\right] + \Pi, \quad r_0 < r < R, \quad \tau_b < \tau < \tau_e; \tag{1}$$

$$t(r,\ \tau_b) = \Phi(r); \tag{2}$$

$$t(R,\ \tau) = t_R(\tau); \tag{3}$$

$$t(r_0,\ \tau) = t_0(\tau), \tag{4}$$

where Γ is a geometric parameter whose value 0, 1, or 2 corresponds to Cartesian, cylindrical, and spherical coordinates, respectively. From Eqs. (1)-(4) we find the initial temperature distribution:

$$t(r,\ \tau_b) = \varphi(r). \tag{5}$$

The solution of the problem (1)-(4) may also be used for determining the instant τ^* at which the given event occurred. In this case the instant τ_b is chosen, e.g., by trial and error from the condition $\tau_b < \tau^*$. Then, as a result of solving the direct heat-conduction problem (1), (3)-(5), we determine τ^*.

For solving retrospective IHCP the method of discrete superposition [4] is used, which proved very successful in solving boundary, coefficient, geometric, and some other inverse problems [5–9]. According to this method the solution of problem (1)–(4) is sought in the form of a model function $t_M(r, \tau)$ which satisfies the heat-conduction equation (1) and the boundary conditions (3), (4) in the entire range under examination. Condition (2) is satisfied only at some set of points whose number is determined in each actual case by proceeding from the required accuracy of the solution and from the error of the input data. Then the solution of the IHCP reduces to seeking some parameters of the system a_j (j = 1, 2, ..., J) which are controlled in accordance with the requirement of coincidence of the model temperature function $t_M(r, \tau)$ with condition (2) at J points of the interval $r_0 < r < R$.

As parameters of the system it is expedient to choose characteristics which determine the function $\varphi(r)$ in condition (5); it suffices to specify it for the examined problem to become a direct heat-conduction problem.

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To express the function $\varphi(r)$ through the controlling parameters a_j, we expand it into some functional series over the coordinate r, e.g., into a Taylor series, and in this series we confine ourselves to some number of first terms:

$$\varphi(r)=b_0+b_1\frac{r-r_0}{R-r_0}+b_2\frac{(r-r_0)(R-r)}{(R-r_0)^2}+b_3\frac{(r-r_0)^2(R-r)}{(R-r_0)^3}+b_4\frac{(r-r_0)^2(R-r)}{(R-r_0)^4}+\ldots \quad (6)$$

Series (6) is characterized by the fact that its terms, from the third one onward, vanish at the bounds of the interval of change of the function $\varphi(r)$. If we proceed from the requirement of agreement of (6) with the boundary conditions (3), (4), we obtain

$$\varphi(r)=t_0(0)+\frac{r-r_0}{R-r_0}t_R(0)+a_1\frac{(r-r_0)(R-r)}{(R-r_0)^2}+ \\ +a_2\frac{(r-r_0)^2(R-r)}{(R-r_0)^3}+a_3\frac{(r-r_0)^2(R-r)}{(R-r_0)^4}+\cdots+a_J\frac{(r-r_0)^{J+1-[(J+1)/2]}(R-r)^{[(J+1)/2]}}{(R-r_0)^{J+1}}, \quad (7)$$

where $[(J + 1)/2]$ is the integral part of the number $(J + 1)/2$. To calculate the parameters of the system — the coefficients a_j ($j = 1, 2, \ldots, J$) of a truncated series representing the function φ — each of them is matched with a certain value of the specified function $\Phi(r_j)$ at the point r_j ($j = 1, 2, \ldots, J$). The points r_j may be chosen arbitrarily but to improve the convergence of series (7) it is expedient to confront the parameter a_1 with the value of the function Φ at the point $r_1 = (r_0 + R)/2$, the parameter a_2 at the point $r_2 = r_0 + 3/4(R - r_0)$, the parameter a_3 at the point $r_3 = r_0 + 1/4\ (R - r_0)$, etc. In the first approximation the values of the parameters a_j are specified arbitrarily, e.g., $a_j = 0$. If we have available the function $\varphi_{(\beta)}(r)$ in the β-th approximation, we can, by solving the system of equations (1), (3)-(5), determine the model temperature function $t_{M(\beta)}$ in the approximation β. For this purpose we may use, e.g., the explicit difference scheme which in the grid $r_i=r_0+ih$, $i=0, 1, \ldots, I$, $h=\frac{R-r_0}{I}$; $\tau_n=nl$, $n=1, 2, \ldots$, may be represented in the following form:

$$t_i^{n+1}=t_i^n+\frac{l}{c\rho(r_0+ih)^\Gamma h^2}\{[\lambda_{i+1}^n(r_0+ih+h)^\Gamma+\lambda_i^n(r_0+ih)^\Gamma]\times \\ \times(t_{i+1}^n-t_i^n)+[\lambda_i^n(r_0+ih)^\Gamma+\lambda_{i-1}^n(r_0+ih-h)^\Gamma](t_i^n-t_{i-1}^n)\}+\frac{l\Pi}{c\rho},\quad i=1, 2, \ldots, I-1; \quad (8)$$

$$t_i^0=\varphi(r_0+ih);\quad t_0^{n+1}=t_0(\tau_{n+1});\quad t_I^{n+1}=t_R(\tau_{n+1}). \quad (9)$$

The difference $\eta_{j(\beta)}=t_{M(\beta)}(r_j, \tau_e)-\Phi(r_j)$, where the model function is determined from Eqs. (8)-(9) with the value $\varphi_{(\beta)}$ in the β-th approximation, is the discrepancy that is used as the signal of mismatch for the subsequent approximations.

The values a_j satisfying the condition

$$|\eta_j|<\delta, \quad (10)$$

where δ is some small positive number, are sought by the method of subsequent minimization of the discrepancies [4]. This method is realized in the following manner. Each parameter is matched with a certain discrepancy η_j, $j = 1, 2, \ldots, J$. A small trial step Δa_1 over the first parameter a_1 is made, and the direct problem is solved; as a result, the increment of the corresponding discrepancy $\Delta\eta_1$ and the approximate value of the derivative $\partial\eta_1/\partial a_1=\Delta\eta_1/\Delta a_1$ are determined. Then several working steps over parameter a_1 are made with fixed values of the other parameters until condition (10) for the discrepancy η_1 (j = 1) is fulfilled. The magnitude of the working steps is determined by the formula

$$\Delta a_j = -A_j \eta_j \frac{\partial a_j}{\partial \eta_j}, \tag{11}$$

in which the coefficient A_j is put equal to unity. The values of the derivative $\partial a_j/\partial \eta_j$ may be refined after each β-th working step, i.e.,

$$\left(\frac{\partial a_j}{\partial \eta_j}\right)_{(\beta)} = \frac{a_{j(\beta)} - a_{j(\beta-1)}}{\eta_{j(\beta)} - \eta_{j(\beta-1)}}. \tag{12}$$

We assume that for the discrepancies η_j, j = 1, 2, ..., s, s < J, corresponding to the parameters $a_1, a_2, \ldots, a_s$, condition (10) is already satisfied, and that we have to determine the values of the parameters $a_1, a_2, \ldots, a_{s+1}$ which ensure that (10) is satisfied for the discrepancies η_j, j = 1, 2, ..., s + 1. At first we make a trial step Δa_{s+1} over the parameter a_{s+1}, and then working steps over this parameter, and the magnitude of the steps is determined by (11). Each step over the parameter a_{s+1} is carried out after a cycle of calculations which is connected with the change of the parameters $a_1, a_2, \ldots, a_s$ until condition (10) for j = 1, 2, ..., s is satisfied. This cycle differs from the cycle of calculations preceding the change of the parameter a_{s+1} by the fact that no trial steps for determining the derivative $\partial\eta_j/\partial a_j$, $j = 1, 2, \ldots, s$ are made. The derivative for the first working step is taken from the preceding cycle, and in this case the increments Δa_j are calculated by formula (11) for $0 < A_j \leqslant 1$. The calculation is discontinued when for all components η_j, j = 1, ..., J of the discrepancy vector condition (10) is fulfilled.

The results of solving various IHCP — boundary, coefficient, geometric, retrospective problems — in unidimensional and two-dimensional statement testify to the efficiency of the described method. The circumstance that the derivatives $\partial\eta_j/\partial a_j$ change very slightly from step to step over parameter a_j makes it possible to attain very quickly (in 2–3 steps) that condition (10) is satisfied.

In processing experimental data, the magnitude of J is determined as the minimum number of points of the interval $[r_0, R]$ which ensures for all the measured values of $\Phi(r_m)$, m = 1, 2, ..., M, M > J that the condition $\|t_M(r_m, \tau_e) - \Phi(\tau_e)\| \leqslant \sigma$ will be fulfilled, where σ is the level of the discrepancy determined by the error of specifying the input data.

To verify the described method of solving retrospective problems, a numerical experiment containing three stages was carried out. At the first stage the direct heat-conduction problem is solved:

$$\frac{\partial t}{\partial \tau} = a \frac{\partial^2 t}{\partial x^2}, \quad 0 < x < L, \quad \tau_b < \tau < \tau_e;$$

$$t(x, \tau_b) = g + g_1 \sin(g_2 x); \quad t(0, \tau) = g_3 + g_4 \tau; \quad t(L, \tau) = g_5 + g_6 \tau^2.$$

At the second stage the IHCP is solved on the assumption that the field $t(x, \tau_b)$ at the instant τ_b is unknown. Instead of this, the temperature distribution $t(x, \tau_e)$ at the instant τ_e, found as a result of solving the direct problem, is specified.

At the third stage the stability of the solution relative to the distortion of the input data is investigated. For this, a distortion in the form of the sinusoidal function $\Delta t(x) = g_7 \sin(g_8 x)$ was superimposed on the temperature field. Figure 1 presents the results of

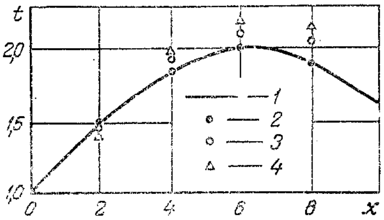

Fig. 1

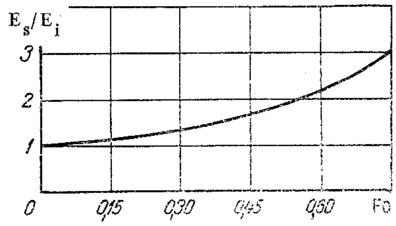

Fig. 2

Fig. 1. Results of calculating the temperature $t(x, \tau_b)$ at the preceding instant: 1) exact value of the function $t(x, \tau_b)$ used in solving the direct problem; 2) solution of the IHCP with the undistorted input data; 3) solution of the IHCP with $g_7 = 0.06$; 4) the same, with $g_7 = 0.10$.

Fig. 2. Dependence of the ratio of the error of the solution to the initial information on the Fourier number.

solving a retrospective IHCP with the following initial data: $g = g_1 = g_3 = 1$; $g_2 = 0.25$; $g_4 = 0.05$; $g_5 = 1.6$; $g_6 = 0.0065$; $g_8 = 0.3$; $\tau_b = 0$; $\tau_e = 0.75$; $a = 1$; $L = 1$; $\delta = 10^{-9}$. Figure 2 shows the change of the ratio of the error of the solution E_s to the error of the initial information E_i in dependence on the number $Fo = a(\tau_e - \tau_b)/L^2$. The considerable increase of E_s/E_i for $Fo > 0.5$ is due to the fact that with increasing Fourier number the influence of the initial data on the temperature field decreases, and with $Fo > 1$ it is quite insignificant.

The realization of one variant of the given problem on a BÉSM-4M computer requires about 10 minutes of computer time. The results of the calculations prove the effectiveness and sufficient accuracy of the described method of solving retrospective problems.

The method of discrete superposition may also be used for finding the temperature field φ at the initial instant in bodies of complex shape. In this case the function φ is expanded according to the degrees of its variables into a series which for two-dimensional bodies has the form

$$\varphi = b_1 + b_2x + b_3y + b_4x^2 + b_5xy + b_6y^2 + \ldots \quad (13)$$

The parameters of the system a_j are chosen on the basis of the series (13) after correlations between its coefficients have been established that are connected with satisfying the specified boundary conditions by (13). The parameters of the system a_j, $j = 1, 2, \ldots, J$, are confronted with the temperatures at the points of the space (x, y, τ) which are determined by the specified additional information.

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QUESTION OF THE CONVERGENCE OF ITERATION METHODS OF SOLVING THE INVERSE HEAT-CONDUCTION PROBLEM

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The convergence of iteration methods of solving the inverse heat-conduction problem depending on the type of desired boundary function is numerically investigated.

Iteration methods of solving inverse boundary-value heat-conduction problems (IHCP) in an extremal formulation are utilized extensively at this time. These methods are based on the search for boundary functions by starting from the requirement of minimization of a certain functional characterizing the measure of the deviation of the calculated temperatures from the temperature measured during the experiment.

Both the density of the heat flux (boundary condition (BC) of the second kind) and the temperature of the surface being heated (BC of the first kind) can be considered as the functions desired.

Fundamental attention is paid in the development of iteration methods to the construction of iterative schemes based on a search for the time dependence of the thermal flux density [1-3]. At the same time, iteration schemes based on the search for the time dependence of the surface temperature have a definite advantage since it is necessary to find a continuous function with a known value at the initial instant t = 0 (the temperature distribution over the thickness is usually known at t = 0). The thermal flux density can hence be determined by conversion of the boundary condition.

To estimate the convergence of iterative methods of solving the IHCP as a function of the kind of desired boundary function, we consider the following inverse problem in the domain $\{0 \leqslant x \leqslant b,\ 0 \leqslant t \leqslant t_p\}$:

$$C(T)\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left[\lambda(T)\frac{\partial T}{\partial x}\right],\ 0 < x < b,\ 0 < t \leqslant t_p, \quad (1)$$

$$T(x,\ 0) = \varphi(x),\ 0 \leqslant x \leqslant b, \quad (2)$$

$$(2-K)\,T(0,\ t) + (1-K)\,\lambda(T)\frac{\partial T(0,\ t)}{\partial x} = u(t), \quad (3)$$

$$-\lambda(T)\frac{\partial T(b,\ t)}{\partial x} = q_2(t), \quad (4)$$

$$T(b,\ t) = f(t), \quad (5)$$

where C(T), $\lambda(T)$, $\varphi(x)$, $q_2(t)$, f(t) are known functions, K is a parameter governing the type of BC on the domain boundary x = 0 (K = 1 is a BC of the first kind and K = 2 of the second kind).

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